NASA/CR-1998-206916
ICASE Report No. 98-8

Transport Coefficients in Rotating Weakly Compressible Turbulence

Robert Rubinstein
ICASE

Ye Zhou
ICASE and IBM

Gordon Erlebacher
Florida State University

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA
Operated by Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contracts NAS1-19480 & NAS1-97046

February 1998

TRANSPORT COEFFICIENTS IN ROTATING WEAKLY COMPRESSIBLE TURBULENCE *

ROBERT RUBINSTEIN†, YE ZHOU‡, AND GORDON ERLEBACHER§

Abstract. Analytical studies of compressible turbulence have found that compressible velocity fluctuations create both effective fluid transport properties and an effective equation of state. This paper investigates the effects of rotation on compressible turbulence. It is shown that rotation modifies the transport properties of compressible turbulence by replacing the turbulence time scale by a rotational time scale, much as rotation modifies the transport properties of incompressible turbulence. But thermal equilibrium properties are modified in a more complex manner. Two regimes are possible: one dominated by incompressible fluctuations, in which the sound speed is modified as it is in non-rotating turbulence, and a rotation dominated regime in which the sound speed enhancement is rotation dependent. The dimensionless parameter which discriminates between regimes is identified. In general, rotation is found to suppress the effects of compressibility. A novel feature of the present analysis is the use of a non-Kolmogorov steady state as the reference state of turbulence. introduction of such steady states expands the power and utility of analytical turbulence closures to a wider range of problems.

Key words. compressible turbulence, rotating turbulence, closure models

Subject classification. Fluid Mechanics

1. Introduction. Rotating weakly compressible turbulence occurs in astrophysical flows and in engineering flows like swirling jets. The development of transport models and sub-grid scale models for such flows is difficult because the coupling between compressibility and rotation introduces at least two natural dimensionless parameters: the turbulent Mach number $M_t = K^{1/2}/c$ and the inverse turbulent Rossby number $\zeta = \Omega K/\varepsilon$. It is not easy to assess the combined effects of these parameters heuristically. Recourse to DNS will require an unrealistically large number of simulations to cover a useful range of parameters. LES will not be practical without sub-grid scale models which are sensitive to both rotation and compressibility: the development of such models is a goal of this paper.

The present work addresses these problems using the rational statistical turbulence closure provided by Yoshizawa's (1986,1995) two-scale direct interaction approximation (TSDIA), which is a natural and analytically tractable extension of Kraichnan's (1959) direct interaction approximation (DIA) to inhomogeneous turbulence. It is based on previous work on the transport theory of weakly compressible turbulence (Rubinstein and Erlebacher, 1997, RE in what follows) and on an analytical theory of rotating turbulence initiated by Zhou (1995).

The problem of weakly compressible turbulence, in which the equations for compressible quantities are linearized about a basic state of incompressible turbulence, was considered from the standpoint of TSDIA in RE. TSDIA leads to a description of compressibility effects on the mean flow equations by a set of trans-

*This research was supported by the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-19480 and NAS1-97046 while the authors were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), Mail Stop 403, NASA Langley Research Center, Hampton, VA 23681-0001.

† Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23681

‡ Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23681 and IBM Research Division, T.J. Watson Research Center, P.O. Box, 218, Yorktown Heights, NY 10598.

§ Program in Computational Science and Engineering, Florida State University, Tallahassee, FL 32306-3075

port coefficients: this description generalizes the eddy viscosity description of incompressible turbulence. Following Yoshizawa, the transport coefficients are expressed in terms of the fundamental statistical descriptors of weakly compressible turbulence: the response and correlation functions of incompressible turbulence (Kraichnan, 1959) and the additional statistical descriptors of the compressible part of the motion which will be defined later in this paper. By assuming that the basic state of incompressible turbulence is in a Kolmogorov steady state, it is possible to derive both formulas for the transport coefficients in terms of single point descriptors of turbulence and, following Yakhot and Orszag (1986), sub-grid scale models for LES of weakly compressible turbulence.

The analysis of rotation effects will be based on Zhou's (1995) scaling theory of rotating turbulence, which demonstrates the existence of a steady state with constant intertial range energy flux in rotating turbulence, but with non-Kolmogorov scaling exponents. In the limit of strong rotation, this theory is closely related to the theory of *weak turbulence* (Zakharov *et al*, 1992) in which nonlinear temporal decorrelation of Fourier modes is dominated by linear dispersive decorrelation. The extension of the theory of RE to rotating turbulence will be accomplished by perturbing about this state of rotating incompressible turbulence instead of about the Kolmogorov state.

Like previous investigations of compressible turbulence (Staroselsky *et al*, 1990; Chandrasekhar, 1951), RE found two kinds of compressibility effects: modified transport coefficients and modified equilibrium properties. Turbulence creates diffusivities for entropy and pressure and enhances the molecular shear and bulk viscosities. But turbulence also modifies the specific heat ratio and creates an effective turbulent pressure. These combined effects lead to an enhanced propagation speed for sound waves in the mean flow; they can be considered as a modified equation of state for compressible turbulence. Rotation alters the transport properties, as it does in incompressible turbulence, by replacing the turbulent time scale by a rotational time scale. It will be shown that rotating compressible turbulence can exhibit two different effective equations of state, depending on the size of the parameter $\zeta^{1/2}M_t$. In general, rotation tends to suppress or counteract the effects of compressibility.

2. Review of the TSDIA Theory of Weakly Compressible Turbulence. The basic result of RE is the following model for weakly compressible turbulence:

$$\frac{\partial S}{\partial t} + \mathbf{U}\cdot\nabla S = \nabla\cdot(\nu^{ss}\nabla S) \tag{1}$$

$$\begin{aligned}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} + \left(\frac{1}{R}-\frac{1}{\hat{R}}\right)\nabla P = \nabla\cdot\{-\frac{2}{3}K\mathbf{I} + \nu[\nabla\mathbf{U}+(\nabla\mathbf{U})^T]^S\}\\ + \nabla(\nu^{uu}\nabla\cdot\mathbf{U})\end{aligned} \tag{2}$$

$$\frac{\partial P}{\partial t} + \mathbf{U}\cdot\nabla P + (\gamma-\hat{\gamma})P\nabla\cdot\mathbf{U} = \nabla\cdot(\nu^{pp}\nabla P) \tag{3}$$

The turbulence dependent properties are defined by the integrals

$$\nu = \frac{4}{15}\int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau\; G(k,\tau)Q(k,\tau) \tag{4}$$

$$\nu^{ss} = \frac{2}{3}\int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau\; G^{ss}(k,\tau)Q(k,\tau) \tag{5}$$

$$\begin{aligned}\nu^{pp} = \frac{2}{3}\int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau G^{pp}(k,\tau)Q(k,\tau)\\ - \frac{1}{c^2}\int_0^\infty 4\pi k^2\; dk \int_0^\infty d\tau\{\frac{2}{3}G^{ww}(k,\tau)+\frac{1}{3}G^{\phi\phi}(k,\tau)\}Q^p(k,\tau)\end{aligned} \tag{6}$$

$$\hat{\gamma} = (\gamma-1)\Pi/P \tag{7}$$

$$\Pi = \gamma \int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau \; kG^{\phi p}(k,\tau) Q^p(k,\tau) \tag{8}$$

$$\nu^{uu} = \frac{1}{3}\int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau \frac{2}{3} G^{ww}(k,\tau)Q(k,\tau) - \frac{\gamma}{R^2c^2}\int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau \; G^{pp}(k,\tau)Q^p(k,\tau) \tag{9}$$

$$\frac{1}{\hat{R}} = \int_0^\infty 4\pi k^2 dk \int_0^\infty d\tau \; kG^{\phi p}(k,\tau)Q(k,\tau) \tag{10}$$

The quantities ν, ν^{ss}, ν^{pp} and ν^{uu} are effective transport properties, whereas the effective turbulent pressure Π and effective mean density $\hat{R}$ define the modified equation of state of compressible turbulence. In Eqs. (4)-(10), G and Q are the response and correlation functions, the fundamental statistical descriptors of isotropic turbulence in DIA (Kraichnan, 1959) and Q^p is the two-time pressure correlation function. The compressible response functions $G^{\phi\phi}, G^{\phi p}, G^{pp}, G^{p\phi}, G^{ww}$ are defined in terms of the normalized Helmholtz decomposition

$$u_i(\mathbf{k},t) = w_i(\mathbf{k},t) + ik_i k^{-1}\phi(\mathbf{k},t) \tag{11}$$

and satisfy the DIA response equations, written in terms of time Fourier transforms,

$$i\omega G^{ww}(\mathbf{k},\omega) + \eta^{ww}G^{ww}(\mathbf{k},\omega) = 1 \tag{12}$$

and

$$\left\{ \begin{bmatrix} -i\omega & k/\rho_0 \\ -\rho_0 c^2 k & -i\omega \end{bmatrix} + \begin{bmatrix} \eta^{\phi\phi} & \eta^{\phi p} \\ \eta^{p\phi} & \eta^{pp} \end{bmatrix} \right\} \begin{bmatrix} G^{\phi\phi} & G^{\phi p} \\ G^{p\phi} & G^{pp} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

Following RE, the response equations will be solved in a Markovianized limit in which the damping functions η in Eqs. (12), (13) are defined by

$$\begin{aligned} \eta^{ww}(\mathbf{k}) = &-\frac{1}{2}P_{ij}(\mathbf{k})\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q})G^{ww}(\mathbf{q},\tau)\times \\ &P_{np}(\mathbf{q})\Gamma^1_{prj}(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mr}(\mathbf{p},\tau) \\ &-\frac{1}{2}P_{ij}(\mathbf{k})\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q})G^{\phi\phi}(\mathbf{q},\tau)\times \\ &P^*_{np}(\mathbf{q})\Gamma^1_{prj}(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mr}(\mathbf{p},\tau) \end{aligned} \tag{14}$$

$$\begin{aligned} \eta^{\phi\phi}(\mathbf{k}) = &-P^*_{ij}(\mathbf{k})\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q})G^{ww}(\mathbf{q},\tau)\times \\ &P_{np}(\mathbf{q})\Gamma^1_{prj}(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mr}(\mathbf{p},\tau) \\ &-P^*_{ij}(\mathbf{k})\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q})G^{\phi\phi}(\mathbf{q},\tau)\times \\ &P^*_{np}(\mathbf{q})\Gamma^1_{prj}(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mr}(\mathbf{p},\tau) \end{aligned} \tag{15}$$

$$\begin{aligned} \eta^{\phi p}(\mathbf{k}) = &\, ik_i k^{-1}\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q})\times \\ &G^{up}_n(\mathbf{q},\tau)\Gamma^2_r(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mr}(\mathbf{p},\tau) \end{aligned} \tag{16}$$

$$\begin{aligned} \eta^{p\phi}(\mathbf{k}) = &-ik_j k^{-1}\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma^2_m(\mathbf{k},\mathbf{p},\mathbf{q})\times \\ &G^{pu}_n(\mathbf{q},\tau)\Gamma^1_{nrj}(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{rm}(\mathbf{p},\tau) \end{aligned} \tag{17}$$

$$\eta^{pp}(\mathbf{k}) = -\int_0^\infty d\tau \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\Gamma_m^2(\mathbf{k},\mathbf{p},\mathbf{q})G^{pp}(\mathbf{q},\tau) \times \Gamma_n^2(\mathbf{q},-\mathbf{p},\mathbf{k})Q_{mn}(\mathbf{p},\tau) \tag{18}$$

where the quantities Γ, which characterize the interactions between the fields, are given by

$$\Gamma^1_{imn}(\mathbf{k},\mathbf{p},\mathbf{q}) = i(q_m\delta_{in} + p_n\delta_{im})$$
$$\Gamma^2_n(\mathbf{k},\mathbf{p},\mathbf{q}) = ik_n$$

These equations are solved approximately in RE by a series expansion in turbulent Mach number. Weak compressibility assumes the existence of a basic state of incompressible turbulence. If this turbulence is in a Kolmogorov steady state, in which isotropy implies that

$$G_{ij}(\mathbf{k},\tau) = G(k,\tau)P_{ij}(\mathbf{k}) \tag{19}$$

$$Q_{ij}(\mathbf{k},\tau) = Q(k,\tau)P_{ij}(\mathbf{k}) \tag{20}$$

the approximation leads, at lowest nontrivial order, to the conclusion that G^{ww} is the response function of a passive vector field, and to the expressions

$$G^{\phi p}(k,\tau) = -\frac{k}{\rho_0\mathcal{S}}\exp(-\eta^{\phi\phi}\tau/2)\sin(\mathcal{S}\tau)H(\tau) \tag{21}$$

$$G^{p\phi}(k,\tau) = \rho_0\frac{c^2k}{\mathcal{S}}\exp(-\eta^{\phi\phi}\tau/2)\sin(\mathcal{S}\tau)H(\tau)$$
$$G^{pp}(k,\tau) = \exp(-\eta^{\phi\phi}\tau/2)\cos(\mathcal{S}\tau)H(\tau) + \frac{\eta^{\phi\phi}}{2\mathcal{S}}\exp\left(-\eta^{\phi\phi}\tau/2\right)\sin(\mathcal{S}\tau)H(\tau)$$
$$G^{\phi\phi}(k,\tau) = \exp(-\eta^{\phi\phi}\tau/2)\cos(\mathcal{S}\tau)H(\tau) - \frac{\eta^{\phi\phi}(k)}{2\mathcal{S}}\exp(-\eta^{\phi\phi}\tau/2)\sin(\mathcal{S}\tau)H(\tau) \tag{22}$$

where

$$\mathcal{S} = \{c^2k^2 - \frac{1}{4}\eta^{\phi\phi}(k)^2\}^{1/2} \tag{23}$$

In non-rotating turbulence, the damping function $\eta^{\phi\phi}$ is determined by the Reynolds analogy

$$\eta^{\phi\phi}(k) = \alpha_\phi\eta(k) \tag{24}$$

where α_ϕ is an inverse Prandtl number, and the incompressible spectral damping function $\eta(k)$ takes the Kolmogorov scaling form

$$\eta(k) = C_D\varepsilon^{1/3}k^{2/3} \tag{25}$$

Analogs of these formulas for rotating turbulence will be developed subsequently. Once such formulas are known, the integrals for transport coefficients Eqs. (4)-(10) can be evaluated explicitly. For turbulence transport models, the integration will extend over the entire inertial range $k \geq k_0$ where k_0 is the inverse integral scale. Subgrid models are derived instead (Yakhot and Orszag, 1986) by integrating only over

Table 1
Scaling results of RE

	Single-point model	General subgrid Model	Smagorinsky model
ν^{pp}	$M_t^2 K^2/\varepsilon$	$\varepsilon\Delta^2/c^2$	$(S^<)^3\Delta^4/c^2$
ν^{uu}	K^2/ε	$\varepsilon^{1/3}\Delta^{4/3}$	$(S^<)\Delta^2$
Π	KM_t^2	$\varepsilon^{4/3}\Delta^{4/3}/c^2$	$(S^<)^4\Delta^4/c^2$
$1/\hat{R}$	M_t^2	$\varepsilon^{2/3}\Delta^{2/3}/c^2$	$(S^<)^2\Delta^2/c^2$

wavevectors which satisfy the condition $k \geq 2\pi/\Delta$, where Δ is the filter size; as usual, this derivation assumes that these scales are in the inertial range. The quantity ε can be closed by Smagorinsky's hypothesis of local energy equilibrium. An interesting alternative, related to postulating the *sweeping hypothesis* for turbulent time correlations, has been suggested recently by Yoshizawa *et al* (1996).

The results of RE are summarized in Table 1. The effective properties are given without the appropriate constants, all of which could in principle be evaluated theoretically. The column labelled "general subgrid model" contains the result of integrating the integral expression over the subgrid scales; the corresponding "Smagorinsky model" is obtained by equating the dissipation rate to the resolved production. Throughout the table, $S^<$ is defined by $(S^<)^2 = S^<_{ij}S^<_{ij}$, where $S^<_{ij}$ is the resolved fluctuating strain rate.

3. Review of the Statistical Theory of Rotating Turbulence. Kolmogorov scaling applies to a homogeneous turbulent steady state with constant energy flux. External effects like rotation will modify this steady state. Rotation is an especially simple external effect, since energy remains an inviscid invariant under rotation, and a steady state with constant energy flux remains possible. Kolmogorov scaling no longer applies to this steady state and the existence of a distinguished time scale, namely the inverse rotation rate, precludes the deduction of the applicable scaling law by dimensional analysis alone. To deduce the scaling, we will appeal to closure in the form of the direct interaction approximation.

For rotating turbulence, the DIA equations of motion take the form

$$\begin{aligned} &\dot{G}_{ij}(\mathbf{k},t,s) + 2P_{ip}(\mathbf{k})\Omega_{pq}G_{qj}(\mathbf{k},t,s) \\ &+ \int_s^t dr\ \eta_{ip}(\mathbf{k},t,r)G_{pj}(\mathbf{k},r,s) = 0 \end{aligned} \tag{26}$$

$$\begin{aligned} &\dot{Q}_{ij}(\mathbf{k},t,s) + 2P_{ip}(\mathbf{k})\Omega_{pq}Q_{qj}(\mathbf{k},t,s) \\ &+ \int_s^t dr\ \eta_{ip}(\mathbf{k},t,r)Q_{pj}(\mathbf{k},r,s) \\ &= \int_0^t dr\ G_{ip}(\mathbf{k},t,r)F_{pj}(\mathbf{k},s,r) \end{aligned} \tag{27}$$

where the eddy damping η and forcing F are defined by

$$\eta_{ir}(\mathbf{k},t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}P_{imn}(\mathbf{k})P_{\mu rs}(\mathbf{p}) \times$$

$$F_{ij}(\mathbf{k},t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}P_{imn}(\mathbf{k})P_{jrs}(\mathbf{k}) \times \begin{array}{l} G_{m\mu}(\mathbf{p},t,s)Q_{ns}(\mathbf{q},t,s) \quad (28) \\ Q_{ns}(\mathbf{p},t,s)Q_{mr}(\mathbf{q},t,s) \quad (29) \end{array}$$

In Eqs. (28) and (29),

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$
$$P_{im}(\mathbf{k}) = \delta_{im} - k^{-2}k_i k_m$$

and Ω_{pq} is the antisymmetric rotation matrix.

The solution of these equations in complete generality is not known. A natural perturbation theory treats the rotation terms as small, and perturbs about an isotropic turbulent state. This approach is adopted by Shimomura and Yoshizawa (1986), who derive a TSDIA theory in which inhomogeneity and rotation are both described by small parameters.

A complementary limit is also of interest. Namely, in the response equation, balance the time derivative by the rotation term, and treat the eddy damping as small. This linear theory of the response equation treats strongly rotating turbulence as a case of *weak turbulence* (Zakharov *et al*, 1992) in which nonlinear decorrelation of Fourier modes is dominated by linear dispersive decorrelation (Waleffe, 1993). The result is conveniently expressed in terms of the *Craya-Herring* basis

$$\mathbf{e}^{(1)}(\mathbf{k}) = \mathbf{k}\times\mathbf{\Omega}/ \mid \mathbf{k}\times\mathbf{\Omega} \mid$$
$$\mathbf{e}^{(2)}(\mathbf{k}) = \mathbf{k}\times(\mathbf{k}\times\mathbf{\Omega})/ \mid \mathbf{k}\times(\mathbf{k}\times\mathbf{\Omega}) \mid$$

or the equivalent helical mode basis used, for example, by Waleffe (1993) and Cambon and Jacquin (1989), and the corresponding tensors

$$\begin{aligned} \xi^0_{ij} &= e^{(1)}_i e^{(2)}_j - e^{(1)}_j e^{(2)}_i \\ \xi^1_{ij} &= e^{(1)}_i e^{(2)}_j + e^{(1)}_j e^{(2)}_i \\ \xi^2_{ij} &= e^{(1)}_i e^{(1)}_j - e^{(2)}_i e^{(2)}_j \\ \xi^3_{ij} &= e^{(1)}_i e^{(1)}_j + e^{(2)}_i e^{(2)}_j \end{aligned} \tag{30}$$

Note that $\xi^3_{ij} = P_{ij}(\mathbf{k})$.

In this approximation, the leading order solution of Eq. (26) is

$$G_{ij}(\mathbf{k},\tau) = \{\cos(\Theta\tau)P_{ij}(\mathbf{k}) + \sin(\Theta\tau)\xi^0_{ij}(\mathbf{k})\}H(\tau) \tag{31}$$

In Eq. (31), $\Theta(\mathbf{k})$ is defined by

$$\Theta(\mathbf{k}) = \mathbf{k}\cdot\mathbf{\Omega}/k$$

$\tau = t - s$ is the time difference, and H is the unit step function.

Let us adopt the *fluctuation-dissipation* hypothesis relating the two-time correlation function to the response function and single-time correlation function

$$Q_{ij}(\mathbf{k},\tau) = G_{im}(\mathbf{k},\tau)Q_{mj}(\mathbf{k}) + G_{jm}(\mathbf{k},-\tau)Q_{mi}(\mathbf{k}) \tag{32}$$

Conditions under which this approximation is reasonable are discussed by Woodruff (1992, 1994). Substituting Eqs. (31) and (32) in Eq. (27) suggests that the single-time correlation function should take the general form containing all of the ξ tensors of Eq. (30)

$$Q_{ij}(\mathbf{k}) = \sum_{0\leq p\leq 3} Q^p(\mathbf{k})\xi^p_{ij}(\mathbf{k}) \tag{33}$$

which is equivalent to the form of the correlation function noted by Cambon and Jacquin (1989). But substituting Eq. (33) in Eq. (32), we find instead that the symmetry

$$Q_{ij}(\mathbf{k},\tau) = Q_{ji}(\mathbf{k},-\tau)$$

leads at lowest order in the weak turbulence approximation to the simpler expression

$$Q_{ij}(\mathbf{k},\tau) = \sin(\Theta\tau)Q(\mathbf{k})\xi^0_{ij}(\mathbf{k}) + \cos(\Theta\tau)Q(\mathbf{k})\xi^3_{ij}(\mathbf{k}) \tag{34}$$

This simplification is a consequence of the approximate treatment of time correlations; refinement of the approximation will recover the full expression Eq. (33).

The DIA inertial range energy balance (Kraichnan, 1971), which states that a steady state with constant energy flux exists, is

$$\begin{aligned} \varepsilon = [I^+ - I^-]P_{imn}(\mathbf{k})\int_0^\infty d\tau\ & 2P_{\mu rs}(\mathbf{p})G_{m\mu}(\mathbf{p},\tau)Q_{ns}(\mathbf{q},\tau)Q_{ir}(\mathbf{k},\tau) \\ & -P_{jrs}(\mathbf{k})G_{ij}(\mathbf{k},\tau)Q_{ns}(\mathbf{p},\tau)Q_{mr}(\mathbf{q},\tau) \end{aligned} \tag{35}$$

where the integration operators in Eq. (35) are defined by

$$\begin{aligned} I^+(k_0) &= \int_{k\geq k_0} d\mathbf{k} \int_{p,q\leq k_0} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \\ I^-(k_0) &= \int_{k\leq k_0} d\mathbf{k} \int_{p,q\geq k_0} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \end{aligned}$$

Substituting Eqs. (31) and (34) in Eq. (35) shows that the time integrals in Eq. (35) will contain the factors

$$T(\mathbf{k},\mathbf{p},\mathbf{q}) = \Omega^{-1}\delta(\pm\mathbf{p}\cdot\boldsymbol{\Omega}/p\Omega \pm \mathbf{q}\cdot\boldsymbol{\Omega}/q\Omega \pm \mathbf{k}\cdot\boldsymbol{\Omega}/k\Omega) \tag{36}$$

where

$$\Omega = (\Omega_{pq}\Omega_{pq})^{1/2}$$

Thus, wave-vector integrations take place over resonant triads only (Waleffe, 1993) and these integrals scale as Ω^{-1} (Zhou, 1995).

Following the derivation of the Kolmogorov steady state, we seek a solution of Eq. (35) of the power-law form

$$Q(\mathbf{k}) = k^{-\alpha-2}f(\boldsymbol{\Omega}\cdot\mathbf{k}/k\Omega) \tag{37}$$

In view of Eq. (36), the energy balance Eq. (35) requires the scaling $\alpha = 2$ and the proportionality $f \sim \sqrt{\varepsilon\Omega}$. Define the shell-averaged energy spectrum

$$E(k) = \oint d\mathbf{k}\ Q(\mathbf{k}) \tag{38}$$

then (Zhou, 1995)

$$E(k) = C_K^{\Omega}\sqrt{\varepsilon\Omega}k^{-2} \tag{39}$$

It is easily verified that the flux integral Eq. (35) converges for an infinite k^{-2} inertial range. Consequently, the scaling law of Eq. (39) is the scaling of a solution of the DIA equations of motion in the weak turbulence approximation.

The anisotropy of rotating turbulence implies that the eddy viscosity is a tensor defined so that

$$< u_i u_j > = \; -\nu_{ijrs}\frac{\partial U_r}{\partial X_s}$$

where TSDIA gives

$$\nu_{ijrs} = \frac{1}{4}\int \; d\mathbf{k} \; \int_0^{\infty} \; d\tau \; G_{ir}(\mathbf{k},\tau)Q_{js}(\mathbf{k},\tau) + (ij)(rs) \tag{40}$$

and the parentheses indicate index symmetrization. Using the general form of the single-time correlation function Eq. (33) and performing the time integration,

$$\nu_{ijrs} = \int d\mathbf{k}\; \frac{1}{\Omega}\delta(\mathbf{k}\cdot\mathbf{\Omega}/k\Omega)\sum_{\lambda,\mu}\xi_{ir}^{\lambda}(\mathbf{k})\xi_{js}^{\mu}(\mathbf{k})Q^{\lambda}(\mathbf{k}) \tag{41}$$

The delta function dependence of the integrand of Eq. (41) implies that this integrand is nonvanishing only for vectors perpendicular to the rotation axis. This fact originates in the resonance condition noted earlier: since TSDIA treats the effect of small turbulent scales on large scales characterizing the mean flow, it evaluates turbulence transport properties in the *distant interaction approximation* (Kraichnan, 1987), in which interactions among modes with wavevectors $\mathbf{k},\mathbf{p},\mathbf{q}$ are all treated as distant interactions for which $k/p, k/q \to 0$. In this limit, only planar triads such that $\mathbf{k}\cdot\mathbf{\Omega} = \mathbf{p}\cdot\mathbf{\Omega} = \mathbf{q}\cdot\mathbf{\Omega} = 0$ can satisfy the resonance condition. This states the "two-dimensionalization" of energy transfer in strongly rotating turbulence, an observation which is consistent with the more comprehensive theory of Mahalov and Zhou (1996).

The isotropic contribution to the eddy viscosity, obtained by multiplying Eq. (41) by the isotropic tensor $\delta_{ij}\delta_{rs} + \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}$, is the scalar

$$\nu = C_{\nu}^{\Omega}\frac{K}{\Omega} \tag{42}$$

For a subgrid-scale model, instead

$$\nu(\Delta) = C_S^{\Omega}\sqrt{\varepsilon/\Omega}\Delta \tag{43}$$

Closing the dissipation rate ε by the resolved production

$$\varepsilon = \frac{1}{2}\nu(\Delta)S_{ij}^{<}S_{ij}^{<} = \nu(\Delta)(S^{<})^2 \tag{44}$$

where $S_{ij}^{<}$ is the resolved fluctuating strain rate, leads to the *Smagorinsky model for strongly rotating turbulence*

$$\nu(\Delta) = C_{S2}^{\Omega}(S^{<})^2\Delta^2/\Omega \tag{45}$$

Thus, the subgrid scale viscosity is suppressed by rotation. The eddy viscosity has the same Δ^2 dependence on grid size as the standard Smagorinsky model for non-rotating turbulence.

Unlike a fully linear theory like rapid distortion theory, the weak turbulence approximation ignores nonlinearity only in the response equation but retains the nonlinearity of the energy flux condition. The corrections due to nonlinearity of the response equation can be systematically evaluated in the weak turbulence approximation. Namely, substituting Eqs. (31) and (39) in Eq. (28) for the eddy damping factor and taking the distant interaction limit,

$$\eta \sim k^2 \int_k^\infty dp\ \frac{1}{\Omega}\sqrt{\varepsilon\Omega}p^{-2} \sim k\sqrt{\varepsilon/\Omega} \tag{46}$$

The corrections to the time scale and energy spectrum therefore have the form

$$\begin{aligned} T &\sim \frac{1}{\Omega}\{1 + O(\Omega^{-3/2})\}^{-1} \\ E &\sim \sqrt{\varepsilon\Omega}k^{-2}\{1 + O((k^2\varepsilon/\Omega^3)^{1/2}\} \end{aligned} \tag{47}$$

The low rotation rate expansion of Shimomura and Yoshizawa (1986) gives the complementary expansions in positive powers of Ω,

$$\begin{aligned} T &\sim \varepsilon^{-1/3}k^{-2/3}\{1 + O(\Omega)\} \\ E &\sim \varepsilon^{2/3}k^{-5/3}\{1 + O(\Omega/\varepsilon^{1/3}k^{2/3}\} \end{aligned} \tag{48}$$

4. The Effect of Rotation on Weakly Compressible Turbulence. In the presence of rotation, centrifugal and Coriolis terms must be added to the momentum equations. However, the effects of rotation on the local thermal equilibrium properties of fluids can usually be neglected, so that the entropy and pressure equations remain unchanged. The modified momentum equations have the form

$$\dot{u}_i + 2\Omega_{ip}u_p - (1 + \frac{\rho}{R})\Omega^2 x_i = \cdots \tag{49}$$

where terms independent of rotation have not been explicitly written. The mean contribution to the centrifugal term should be added to the mean momentum equation. The fluctuating component has both solenoidal and potential parts, which will appear in the equations for w_i and ϕ.

The Coriolis term must be treated by the Helmholtz decomposition Eq. (12) followed by a second decomposition into solenoidal and potential parts,

$$\begin{aligned} \Omega_{ip}u_p &= [P_{ij}(\mathbf{k}) + P^*_{ij}(\mathbf{k})]\Omega_{jp}[\mathbf{u}^\infty_p + w_p + ik^{-1}k_p\phi] \\ &= P_{ij}(\mathbf{k})\Omega_{jp}\mathbf{u}^\infty_p + P_{ip}\Omega_{pj}w_p + ik^{-1}\Omega_{ip}k_p\phi \\ &+ P^*_{ip}\Omega_{jp}\mathbf{u}^\infty_p + P^*_{ip}\Omega_{pj}w_p \end{aligned} \tag{50}$$

The first term in Eq. (50) is the usual Coriolis term which causes the modified statistically steady state described in Sect. 3. The second term is the analogous Coriolis term for the compressible field w_i. The third term is transverse; therefore, it couples the field ϕ to the w_i equation. This coupling does not occur in non-rotating turbulence. The fourth term shows that the incompressible field is a source for the compressible potential field ϕ. Finally, the fifth term couples the field w_i to the ϕ equation. The anisotropy introduced by rotation makes this type of coupling between a scalar and a vector possible; this coupling is obviously impossible in isotropic turbulence. Note the important fact that ϕ is not coupled linearly to itself through rotation.

In the analysis of non-rotating weakly compressible turbulence, it was not possible to treat all couplings exactly; likewise, the present analysis will treat all rotation-dependent couplings perturbatively, except for the rotational self-couplings of the solenoidal fields w and $\mathbf{u}^\infty$. This defines the *diagonal approximation* for rotational effects

$$\begin{aligned}\dot{\mathbf{u}}_i^\infty + 2P_{ij}\Omega_{jp}\mathbf{u}_p^\infty &= \cdots \\ \dot{w}_i + 2P_{ij}\Omega_{jp}w_p &= -ik^{-1}\Omega_{ip}k_p\phi + \cdots \\ \dot{\phi} &= -2P_{ip}^*\Omega_{jp}(\mathbf{u}_p^\infty + w_p)\cdots \end{aligned} \tag{51}$$

where dots indicate centrifugal and rotation-independent terms. In this approximation, it is appropriate to replace the Kolmogorov steady state for the basic incompressible field $\mathbf{u}^\infty$ by the k^{-2} steady state for strong rotation described in Sect. 3.

5. Evaluation of the Transport Coefficients.

5.1. The entropy diffusivity. Since the entropy equation is not altered by rotation, G^{ss} remains the response function of a passive scalar. The usual Reynolds analogy, which postulates the proportionality of the time correlations of the velocity and passive scalar fields leads, following the derivation of Eq. (42) to

$$\nu^{ss} = C_{ss}^{\Omega}\frac{K}{\Omega} \tag{52}$$

in the strong rotation limit.

5.2. The quantities Π and $\hat{R}$. According to Eqs. (8) and (10), the evaluation of these quantities requires the response function $G^{\phi p}$ and the two-time pressure correlation in rotating incompressible turbulence. The perturbation theory of RE begins by observing that G^{ww} is the response function of a passive vector field. Whereas this could be asserted exactly in non-rotating turbulence, the couplings which occur in Eq. (51) restrict this assertion to the lowest order in the diagonal approximation for rotation effects.

Thus, we assume that to lowest order G^{ww} takes the same form as the response function in Eq. (31)

$$G_{ij}^{ww}(\mathbf{k},\tau) = \{\cos(\Theta\tau)P_{ij}(\mathbf{k}) + \sin(\Theta\tau)\xi_{ij}^0(\mathbf{k})\}H(\tau) \tag{53}$$

As in RE, the damping functions except for $\eta^{\phi\phi}$ all vanish to this order. We must first check that the integral in Eq. (15) which defines this function converges for the k^{-2} spectral scaling. As usual, this convergence check requires the evaluation of the geometric factors in this integral in the limits $p, q \to \infty$. This factor has the form

$$\begin{aligned}&(q_m\delta_{in} + k_m\delta_{in})(k_r\delta_{jp} - p_j\delta_{pr})\xi_{mr}^\lambda(\mathbf{p})\xi_{np}^\mu(\mathbf{q}) \\ &= k_mk_r\xi_{mr}^\lambda(\mathbf{p})\xi_{ij}^\mu(\mathbf{q}) - k_mp_j\xi_{mp}^\lambda(\mathbf{p})\xi_{ip}^\mu(\mathbf{q}) \\ &+ k_nk_r\xi_{ir}^\lambda(\mathbf{p})\xi_{nj}^\mu(\mathbf{q}) - k_np_j\xi_{ip}^\lambda(\mathbf{p})\xi_{np}^\mu(\mathbf{q})\end{aligned} \tag{54}$$

The second and fourth terms, which power counting would suggest cause logarithmic divergence in Eq. (15), instead vanish by symmetry. The remaining terms are of order p^{-2} and therefore converge for large p. The infrared divergence when $p \to 0$ is explained by the sweeping effect on Eulerian time correlations to which Kraichnan (1964) first called attention. Rather than evaluate the transport coefficients in a Lagrangian statistical theory (Kraichnan, 1965; Kaneda, 1981) in which this divergence does not occur, we shall simply regularize the Eulerian theory by integrating over the range $p, q \geq k$ only (Kraichnan, 1963). This regularization leads finally to the result

$$\eta^{\phi\phi}(\mathbf{k},\tau) = C_{\phi\phi}^{\Omega}\sqrt{\varepsilon/\Omega}k \tag{55}$$

which can be compared to the correction term computed in Eq. (46).

The evaluation of the two-time pressure correlation function is actually simpler in the weak turbulence approximation for rotating turbulence that it is for non-rotating turbulence because the resonance condition and even time parity of time correlation functions force the time dependence $\cos(\Theta\tau)$. As in the non-rotating case, convergence of the single-time correlation under the quasi-normal approximation appropriate to the direct interaction approximation is easily established, and we conclude that

$$Q^p(k,\tau) = C_{B,\Omega}\cos(\Theta\tau)\varepsilon\Omega k^{-3}/4\pi k^2 \tag{56}$$

It is now straightforward to verify the calculation

$$\begin{aligned}\Pi/\gamma &= \frac{1}{2}\int d\mathbf{k}\int_0^\infty d\tau\ \exp(-\sqrt{\theta}k\tau)\times \\ &\quad [\sin(Vk+\Omega)\tau + \sin(Vk-\Omega)\tau]Q^p(k) \\ &= \int_0^\infty 4\pi dk\ 2\frac{\varepsilon\Omega}{V}k^{-2}\frac{Vk^3(\theta)+V^2)-Vk\Omega^2}{[\theta+V^2]^2k^4+2(\theta-V^2)\Omega^2k^2+\Omega^4}\end{aligned} \tag{57}$$

where

$$V = \sqrt{c^2-\theta}$$

and

$$\theta = C^{\Omega}_{\phi\phi}\frac{\varepsilon}{\Omega}$$

The integral in Eq. (57) could be evaluated in closed form. However, its limits for large and small wavenumber k are easily evaluated directly. Assume that the k^{-2} spectrum will be cut off at wavenumber k_0, which is either the inverse integral scale, or inverse filter size. If the cutoff scale k_0 is large, then approximately

$$\Pi/\gamma = 4\pi\int_{k_0}^\infty C(Vk)^{-1}\frac{1}{c}\varepsilon\Omega k^{-2} = C'\frac{\varepsilon\Omega}{c^2}k_0^{-2} \tag{58}$$

Setting k_0 to the inverse integral scale leads to the single-point model

$$\Pi/\gamma = C^{\Omega}_{\Pi>}\frac{K^2}{c^2} \tag{59}$$

These results coincide with those of RE except that the energy spectrum and total kinetic energy are determined from the k^{-2} spectrum instead of from the Kolmogorov spectrum. However, the large scales contribute a different limit, namely,

$$\Pi/\gamma = 4\pi\int_{k_0}^{k^*}\frac{Vk}{c\Omega^2}\varepsilon\Omega K^{-2} = C^{\Omega}_{\Pi<}\frac{\varepsilon}{\Omega}\log(k^*/k_0) \tag{60}$$

The scale k^* is the scale at which the high wavenumber limit Eq. (58) applies; its value is calculated below. Note the logarithmic contribution.

The approximation Eq. (58) applies to small scales which are relatively insensitive to rotation. Whereas the theory of weak compressibility requires the condition $K << c^2$, the limit Eq. (58) requires the much stronger condition

$$K << c\sqrt{\varepsilon/\Omega} \tag{61}$$

At very high rotation rates, a range of large scales such that $\Omega >> ck$ can exist. Such scales are described instead by the limit Eq. (60). The transition between spectral regions is determined by the parameter

$$k^* = \frac{\Omega}{c} \tag{62}$$

Since Eq. (60) describes a condition in which compressibility effects have been reduced, we conclude that rotation tends to counteract the effects of compressibility. The single-point parameter which discriminates between regimes is the ratio

$$\frac{K}{c\sqrt{\varepsilon/\Omega}} = \frac{K^{1/2}}{c}\sqrt{\Omega K/\varepsilon} = \zeta^{1/2} M_t \tag{63}$$

Weak compressibility is defined by the condition that M_t is small. When in addition $\zeta^{1/2}M_t$ is small, rotation effects are weak; strong rotation effects occur only if $\zeta^{1/2}M_t$ is large. Note that since rotation reduces the energy transfer ε, large values of the parameter ζ are possible in rotating compressible turbulence.

The evaluation of the coefficient $\hat{R}$ proceeds similarly. However, it should be noted that the anisotropy of rotating turbulence actually makes this quantity a tensor, so that the mean equations of motion contain the pressure term

$$[\frac{1}{R}\delta_{ij} - \frac{1}{\hat{R}}\alpha_{ij}]\frac{\partial P}{\partial X_j}$$

instead of the isotropic pressure term of Eq. (3). The possibility that sound waves in the mean velocity field become dispersive as a result of the direction dependence of the effective sound speed deserves further investigation. In this article, the sound speed will be treated as a scalar. By comparing Eqs. (8) and (10), it is evident that $\hat{R}$ is found from the previous calculation by multiplying the integrand in Eq. (8) by $k\sqrt{\varepsilon/\Omega}$. The small scale limit becomes

$$\frac{1}{\hat{R}} = C^{\Omega}_{R>}\frac{\varepsilon\Omega k_0^{-2}}{c^2}\frac{k_0}{\sqrt{\varepsilon\Omega}} = C^{\Omega}_{R>}\frac{\sqrt{\varepsilon\Omega}k_0^{-1}}{c^2} \tag{64}$$

and the large scale limit becomes

$$\frac{1}{\hat{R}} = C^{\Omega}_{R<}\frac{\varepsilon}{\Omega}\frac{k_0}{\sqrt{\varepsilon\Omega}} = C^{\Omega}_{R<}\sqrt{\varepsilon/\Omega^3}k_0 \tag{65}$$

The single-point quantities corresponding to Eqs. (64) and (65) are simply

$$\frac{1}{\hat{R}} = \begin{cases} M_t^2 & \text{for } k_0 \to \infty \\ 1/\zeta & \text{for } k_0 \to 0 \end{cases} \tag{66}$$

Eq. (2) shows that $\hat{R}$ represents a modification of the mean density due to compressible flucations. Eqs. (65) and (66) show that this modified mean density actually vanishes for very large rotation rates, again suggesting that rotation tends to counteract compressibility effects.

5.3. The pressure and bulk viscosities. Like $\hat{R}$, these quantities are tensors in rotating turbulence, but only scalar values will be computed here. Eq. (6) shows that the pressure diffusivity ν^{pp} is the sum of two terms. For the first term, in the low turbulent Mach number limit,

$$\begin{aligned} \nu_1^{pp} &= \int_{k_0}^{\infty} dk\ \frac{\eta^3/4 + 2\eta\mathcal{S}^2}{(\eta/2)^4 + 2(\eta/2)^2(\mathcal{S}^2+\Omega^2) + (\mathcal{S}^2-\Omega^2)^2}\sqrt{\varepsilon\Omega}k^{-2} \\ &\sim \int_{k_0}^{\infty} dk\ \sqrt{\varepsilon/\Omega}(c^2 - 3\theta/4)^{-1}k^{-1}\sqrt{\varepsilon\Omega}k^{-2} \\ &= C_1\frac{K^2}{\Omega(c^2-3\theta/4)} \end{aligned} \tag{67}$$

Table 2
Main Scaling Results

	Single-point model	General subgrid Model	Smagorinsky model
ν^{pp}	$M_t^2 K/\Omega$	$\varepsilon\Delta^2/c^2$	$(S^<)^3\Delta^4/c^2$
ν^{uu}	K/Ω	$\sqrt{\varepsilon/\Omega}\Delta$	$(S^<)^2\Delta^2/\Omega$
Π (large k)	KM_t^2	$\varepsilon\Omega\Delta^2/c^2$	$\Omega(S^<)^3\Delta^4/c^2$
Π (small k)	ε/Ω	$\varepsilon\Delta^0/\Omega$	$(S^<)^3\Delta^2/\Omega$
$1/\hat{R}$ (large k)	M_t^2	$\sqrt{\varepsilon\Omega}\Delta/c^2$	$(S^<)^{3/2}\Omega^{1/2}\Delta^2/c^2$
$1/\hat{R}$ (small k)	$\varepsilon/K\Omega$	$\sqrt{\varepsilon/\Omega^3}\Delta^{-1}$	$(S^<)^{3/2}\Omega^{-3/2}\Delta^0$

where η is the incompressible damping factor of Eq. (25) This expression applies to both large and small values of the cutoff scale k_0. The second term is evaluated like the entropy diffusivity as

$$\nu_2^{pp} = \frac{\varepsilon\Omega}{c^2}k_0^{-2} = C_2\frac{K^2}{\Omega c^2} \tag{68}$$

In summing these contributions, care must be taken that cancellations between these terms do not change the limits of the transport coefficients. In the absence of such cancellations,

$$\nu^{pp} = C_{pp}^{\Omega}\frac{K}{c^2}\frac{K}{\Omega} \tag{69}$$

This result can be compared with the non-rotating result in RE. Rotation has the same effect that it has on the incompressible eddy viscosity: the turbulence time scale K/ε is replaced by $1/\Omega$.

A similar calculation leads to an explicit expression for the bulk viscosity,

$$\nu^{uu} = C_{\Omega}^{uu}\frac{K}{\Omega} \tag{70}$$

Note that in RE, this quantity was of order M_t^0. Again, in the limit of strong rotation, the consequence of rotation is the simple replacement of the incompressible turbulence time scale by $1/\Omega$.

The results of this section are summarized in Table 2. The format follows Table 1, except that both large and small scale limits are given for the equilibrium properties Π and $\hat{R}$.

6. The Equation of State of Rotating Weakly Compressible Turbulence. Beginning with the first mixing length models, turbulence modeling has always postulated that turbulent fluctuations modify fluid transport properties by replacing thermal fluctuations with scale dependent macroscopic turbulent fluctuations. Analyses of compressible turbulence (Chandrasekhar, 1952, Staroselsky *et al*, 1990) have noted analogous turbulence effects on thermal equilibrium properties: velocity fluctuations lead to a modified equation of state of compressible turbulence. Thus, Eqs. (6), (8) and (10) show that fluctuations modify γ and lead to changes of the effective pressure and density represented by the terms Π and $\hat{R}$.

In rotating compressible turbulence, the picture is somewhat more complex. The transport coefficients ν are all modified like the incompressible eddy viscosity: the turbulence time scale K/ε is replaced by the rotation time scale $1/\Omega$. But thermal equilibrium properties are modified so that two different regimes are possible: a regime dominated by incompressible fluctuations, in which the sound speed is modified as it is in non-rotating turbulence, and a rotation dominated regime in which the sound speed enhancement is proportional to $\sqrt{\varepsilon/\Omega}$ instead. As noted earlier, this limit indicates a reduction of compressibility effects on large scales by rotation. Pure compressibility effects enter the evolution equations through the turbulent transport and thermal equilibrium properties of Eqs. (2)- (4) which are evaluated in Table 1; the further effects of rotation on the evolution equation appear through the rotation-induced modifications of the transport and equilibrium properties which are evaluated in Table 2.

7. Conclusions. Yoshizawa's TSDIA formalism provides a systematic approach to the complex coupled field problem of rotating weakly compressible turbulence. In this problem, the appearance of two natural dimensionless parameters would frustrate attempts to derive appropriate models heuristically. In particular, the modified equation of state in rotating weakly compressible turbulence is not obvious.

The main defect of this calculation is the assumption of a diagonal approximation for rotation effects, although like RE, we do not make a diagonal approximation for compressibility effects. We should attempt either to verify that the non-diagonal terms are small, or look for a more sophisticated approximation.

A novel feature of the present analysis is the use of a non-Kolmogorov steady state as the reference state of turbulence in a TSDIA calculation. The introduction of such steady states expands the power and utility of TSDIA to a wider range of problems.

Compressible swirling jets are a natural class of flows to which the present theory should apply. The theory is consistent with the general observation that rotation counteracts the suppression of mixing which is the physically most prominent feature of compressible jets. But quantitative conclusions will obviously require computational tests of the theory.

REFERENCES

[1] Cambon, C., and Jacquin, L., 1989, *Spectral approach to non-isotropic turbulence subjected to rotation,* J. Fluid Mech., Vol. 202, p. 295.

[2] Chandrasekhar, S., 1951. *The fluctuation of density in isotropic turbulence,* Proc. Roy. Soc. London A vol. 210, p. 18.

[3] Kaneda, Y., 1981, *Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function,* J. Fluid Mech., Vol. 107, p. 131.

[4] Kraichnan, 1959. *The structure of turbulence at very high Reynolds number,* J. Fluid Mech., vol. 5, p. 497.

[5] Kraichnan, 1964. *Kolmogorov's hypotheses and Eulerian turbulence theory,* Phys. Fluids, vol. 7, p. 1723.

[6] Kraichnan, R. H., 1965, *Lagrangian-history closure approximation for turbulence,* Phys. Fluids Vol. 8, p. 575.

[7] Kraichnan, R. H., 1971, *Inertial range transfer in two and three dimensional turbulence,* J. Fluid Mech., Vol. 47, p. 525.

[8] Kraichnan, R.H., 1987. *An interpretation of the Yakhot-Orszag turbulence theory,* Phys. Fluids, vol. 30, 2400.

[9] Mahalov, A., and Zhou, Y., 1996 *Analytical and phenomenological studies of rotating turbulence ,* Phys. Fluids, Vol. 8, p. 2138.

[10] Rubinstein, R., and Erlebacher, E., 1997 *Transport coefficients in weakly compressible turbulence,* Phys. Fluids, Vol. 9.

[11] Shimomura, Y., and Yoshizawa, A., 1986 *Statistical analysis of anisotropic turbulent viscosity in a rotating system,* J. Phys. Soc. Japan, Vol. 55, p. 1904.

[12] Staroselski, I., Yakhot, V., Orszag, S.A., and Kida, S., 1990. *Long time large scale properties of a randomly stirred compressible fluid,* Phys. Rev. Lett. vol. 65, p. 171.

[13] Waleffe, F., 1993, *Inertial transfers in the helical decomposition,* Phys. Fluids, Vol. 5, p. 677.

[14] Woodruff, S.L., 1992, *Dyson equation analysis of inertial-range turbulence,* Phys. Fluids A, Vol. 5, p. 1077.

[15] Woodruff, S.L., 1994. *A similarity solution for the direct interaction approximation and its relationship to renormalization group analyses of turbulence,* Phys. Fluids, vol. 6, p. 3051.

[16] Yakhot, V., and Orszag, S. A., 1986, *Renormalization group analysis of turbulence* J. Sci. Comput., Vol. 1, p. 3.

[17] Yoshizawa, A., 1984, *Statistical analysis of the deviation of the Reynolds stress from its eddy viscosity representation,* Phys. Fluids, Vol. 27, p. 3177.

[18] Yoshizawa, A., 1995. *Simplified statistical approach to complex turbulent flows and ensemble-mean compressible turbulence modeling,* Phys. Fluids, vol. 7, p. 3105.

[19] Yoshizawa, A., Tsubokura, M., Kobayashi, T., and Taniguchi, M., 1996. *Modeling of the dynamic subgrid scale viscosity in large eddy simulation,* Phys. Fluids, vol. 8, p. 2254.

[20] Zakharov, V. E., L'vov, V. S., and Falkovich, G., 1992, *Kolmogorov Spectra of Turbulence* I, Springer.

[21] Zhou, Y., 1995, *A phenomenological treatment of rotating turbulence,* Phys. Fluids, Vol. 7, p. 2092.